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SMALL OSCILLATIONS AND SPHERICAL MOTION OF A GYROSTAT IN PSEUDO-EUCLIDEAN SPACE

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The method known in the theory of linear oscillations for analyzing degenerate dynamic systems is used here for establishing particular cases of spherical motion of a heavy gyrostat in a three-dimensional pseudo-Euclidean space for which exists a linear invariant of equations of motion.

The problem of spherical motion of a solid body with cavity completely filled by a homogeneous incompressible fluid was considered by Joukowski [1]. Poincaré [2] established the interrelationship between the problem of finding supplementary integrals of the Hamiltonian system and the degeneration phenomenon similar to that of the degeneration of the resonance kind in linear oscillations. His idea is used in [3, 4] for interpreting particular cases of spherical motion of a heavy solid body for which Euler's dynamic equations have a linear invariant. This idea is based on the argument that small oscillations of a solid body in the neighborhood of its stable equilibrium position may be defined by the first terms of expansion of integrals of Euler's nonlinear dynamic equations. In that case the initial conditions must not contain restrictions which exclude small oscillations of the solid body in the neighborhood of that position. It is shown in [2] that resonance at small oscillations of a system of linked oscillators indicates the possibility of existence of supplementary integrals of its dynamic equations. With certain restrictions imposed on the geometry of the body mass and on initial parameters, the supplementary integrals of the linear problem of small oscillations may simultaneously be the integrals of Euler's nonlinear dynamic equations.

The described above method is extended here to the spherical motion of a gyrostat in a uniform gravity field of the pseudo-Euclidean space ${}^1R_3$ whose metric tensor $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ has the components $g_{11} = g_{22} = -1$, $g_{33} = 1$ and $g_{ij} = 0$ for $i \neq j$, where $\mathbf{e}_i$ are unit vectors of some basic reference point of the considered space (*). In what follows the definition of a gyrostat in a pseudo-Euclidean space ${}^1R_3$ is that of a gyrostat located inside an isotropic cone of that space, and the vertex of that cone being the fixed point O. The radius vectors of the gyrostat points are in this case eigenvectors. Concepts of the kinetic moment and of moments of inertia of a solid body about nonisotropic axes, and analogs of Euler's angles were introduced in [6, 7] on the assumption of the validity of fundamental axioms of classical dynamics in the pseudo-Euclidean space (**).

*) The notation ${}^lR_n$ was introduced by Rozenfel'd [5].
**) See also Kosogliad, E. I., On the dynamics of a solid body in a pseudo-Euclidean space and in the Lobacevski plane. Dissertation, Kazan' Univ., 1970.

1. Assuming the validity of axioms and principles of classical dynamics in the pseudo-Euclidean space ${}^1R_3$ we obtain analogs of Joukowski's equations [1] for a gyrostat in the axes of basic frame of reference $S\ (O,\ x,\ y,\ z)$ of that space with the basis $\mathbf{e}_x,\ \mathbf{e}_y,\ \mathbf{e}_z$, where the x- and y-axes are imaginary and z is the eigenaxis.

Let

$$\mathbf{G} = \mathbf{I}\cdot\boldsymbol{\omega} + \boldsymbol{\lambda}, \quad \mathbf{g} = g\boldsymbol{\gamma}, \quad \mathbf{l} = M\ \mathbf{r}_c$$

where $\mathbf{G}$ and $\boldsymbol{\lambda}$ are, respectively, the kinetic moment of the gyrostat and the hydrostatic moment relative to point O ($\boldsymbol{\lambda} = \text{const}$ in S); $\mathbf{I}$ is the inertia tensor of the body transformed according to Joukowski, constructed for point O whose principal components are A, B and C; $\boldsymbol{\omega}\ (p,\ q,\ r)$ is the instantaneous absolute angular velocity of the basic frame of reference S; $\boldsymbol{\gamma}$ is the direction unit vector of the uniform gravity force field of intensity g, and M and $\mathbf{r}_c$ are the mass and the radius vector of the gyrostat mass.

We use the representation

$$\mathbf{G} = (Ap + \lambda_x)\mathbf{e}_x + (Bq + \lambda_y)\mathbf{e}_y - (Cr + \lambda_z)\,\mathbf{e}_z$$

and apply the theorem about the variation of the gyrostat kinetic moment relative to point O. Using the definition of the vector product of vectors in space ${}^1R_3$ given in [7], we obtain

$$\begin{aligned} &Ap^{\cdot} + (C + B)qr + \lambda_z q + \lambda_y r = l_z\gamma_y - l_y\gamma_z \\ &Bq^{\cdot} - (A + C)rp - \lambda_x r - \lambda_z p = l_x\gamma_z - l_z\gamma_x \\ &Cr^{\cdot} - (B - A)pq - \lambda_y p + \lambda_x q = l_y\gamma_x - l_x\gamma_y \end{aligned} \tag{1.1}$$

The Poisson equations are of the form [7]

$$\gamma_x^{\cdot} = q\gamma_z - r\gamma_y, \quad \gamma_y^{\cdot} = r\gamma_x - p\gamma_z, \quad \gamma_z^{\cdot} = q\gamma_x - p\gamma_y \tag{1.2}$$

with $\boldsymbol{\gamma}^2 = -\gamma_x^2 - \gamma_y^2 + \gamma_z^2 = \varkappa$, where $\varkappa = 1,\ -1,\ 0$ when the gravity force vector is either an eigenvector, or an imaginary or isotropic vector, respectively. We specify conditions

$$y_c = \lambda_y = 0 \tag{1.3}$$

The stable equilibrium position of the transformed body with condition (1.3) in the specified field is defined by formula

$$\begin{aligned} &\gamma_y^{\circ} = 0, \quad l_x\gamma_z^{\circ} - l_z\gamma_x^{\circ} = 0, \quad l_z\gamma_z^{\circ} - l_x\gamma_x^{\circ} = l \\ &r_c^2 = z_c^2 - x_c^2 \end{aligned} \tag{1.4}$$

Henceforth the gyrostat parameters related to the specified position will be defined by a zero superscript.

Let p, q and r be quantities of the first order of smallness and the unit vector $\boldsymbol{\gamma}$ to differ slightly from $\boldsymbol{\gamma}^{\circ}$. The linearized system (1.1) with conditions (1.3) and (1.4) then assumes the form

$$\begin{aligned} &p^{\cdot\cdot} + a_{11}p + a_{13}r = 0, \qquad q^{\cdot\cdot} + a_{22}q = 0 \\ &r^{\cdot\cdot} + a_{31}p + a_{33}r = 0 \\ &a_{11} = A^{-1}(B^{-1}\lambda_z^2 + l_z\gamma_z^{\circ}), \quad a_{13} = A^{-1}(B^{-1}\lambda_x\lambda_z - l_z\gamma_x^{\circ}) \\ &a_{22} = B^{-1}(C^{-1}\lambda_x^2 + A^{-1}\lambda_z^2 + l) \\ &a_{31} = C^{-1}(B^{-1}\lambda_x\lambda_z - l_x\gamma_z^{\circ}), \quad a_{33} = C^{-1}(B^{-1}\lambda_x^2 + l_x\gamma_x^{\circ}) \\ &(a_{11}^2 + a_{31}^2 \neq 0) \end{aligned} \tag{1.5}$$

System (1.5) can be taken as representing the first (linear) approximation equations in the initial point neighborhood.

Let us assume that in the ${}^1R_3$ space vectors λ and γ° are collinear. Then $a_{13}a_{31} = a_{11}a_{33}$, which is equivalent to the condition

$$\lambda_x\gamma_z{}^\circ + \lambda_z\gamma_x{}^\circ = 0 \qquad (1.6)$$

The linear transformation

$$P = -a_{31}p + a_{11}r, \quad Q = q, \quad R = a_{11}p + a_{13}r \qquad (1.7)$$

according to condition (1.6) transforms system (1.5) to the following:

$$P^{\cdot\cdot} = 0, \quad Q^{\cdot\cdot} + \omega_2{}^2Q = 0, \quad R^{\cdot\cdot} + \omega_3{}^2R = 0 \qquad (1.8)$$
$$\omega_2{}^2 = a_{22}, \quad \omega_3{}^2 = a_{11} + a_{33}$$

Equations (1.8) are of the form of equations for longitudinal oscillations of linked oscillators with principal frequencies $\omega_1=0$, ω_2 and ω_3 [8, 9]. Note that degeneration of a similar character occurs in the problem of torsional oscillations of a shaft with rigidly mounted disks [10] (see also [11]).

Let us now consider the phase representation of motion. Note that the linear transformation (1.7) does not alter the topological structure of the phase plane. Equations (1.8) imply that the representing point in the phase plane ξ, η of space ${}^1R_3$ moves along the elliptic circumference

$$\xi^2 + \eta^2 = \sigma^2$$
$$\xi = \left\| \begin{matrix} \omega_2 Q \\ \omega_3 R \end{matrix} \right\|, \quad \eta = \left\| \begin{matrix} Q^\cdot \\ R^\cdot \end{matrix} \right\|, \quad \sigma^2 = \xi_0{}^2 + \eta_0{}^2$$

Henceforth the zero subscript denotes parameters at $t = 0$.

Let us consider the integral $P^\cdot = 0$ which follows from the first equation of system (1.8), in which in accordance with the linearized system (1.1) and conditions (1.3), (1.4) and (1.6). It follows directly from conditions (1.4) and (1.6) that

$$\lambda_x l_z + \lambda_z l_x = 0 \qquad (1.9)$$

This condition defines the collinearity of vectors λ and $\mathbf{l}$ in the space ${}^1R_3$. Thus, if vectors λ and γ° are collinear and conditions (1.3) and (1.4) are satisfied, vectors λ and $\mathbf{l}$ are also collinear. Condition (1.9) is one of the relationships that define the gyrostatic analog of the Hesse-Appelrot case in the space ${}^1R_3$.

The integral $P^\cdot = 0$ yields for Eqs. (1.5) the invariant

$$-a_{31}p + a_{11}r = m \qquad (1.10)$$

where m is the constant of integration. This corresponds to the statement that the linear invariant of system (1.5) exists when the characteristic equation

$$\nu^3 - (\omega_2 + \omega_3)\nu^2 + \omega_2\omega_3\nu = 0$$

of the latter has the root $\omega_1 = 0$ [12].

Let us derive the condition for which together with the stipulated assumptions the linear invariant (1.10) of the linearized system (1.5) is also the invariant of the nonlinear system (1.1). Taking into account (1.9) from the condition $P^\cdot = 0$, for Eqs. (1.1) we obtain

$$[A\,(B - A)a_{11}p + C\,(C + B)a_{31}r]q = 0 \qquad (1.11)$$

Formula (1.11) (with conditions (1.3)) represents a particular case of equations of the Staude cone in the ${}^1R_3$ space, whose generatrices are the axes of permanent rotations of the heavy gyrostat.

If $q(t) \neq 0$, the invariant (1.10) with condition (1.11) can be represented in the form

$$[A(B-A)a_{11}^2 + C(C+B)a_{31}^2]p = \text{const}$$

or, if we exclude the case $p(t) = \text{const}$, by

$$C(B-A)(\lambda_z^2 + Bl_z\gamma_z^\circ)^2 + A(C+B)(\lambda_x\lambda_z - Bl_x\gamma_z^\circ)^2 = 0$$

hence, in accordance with (1.9), we have

$$C(B-A)z_c^2 + A(C+B)x_c^2 = 0 \tag{1.12}$$

This condition, together with the constraint $y_c = 0$ (1.3), is the analog of the Hesse-Appelrot case of a solid body in the ${}^1R_3$ space [13]. Note that when conditions (1.12) and (1.9) are satisfied, $m = 0$. Hence the invariant (1.10) assumes the form [13]

$$Ax_cp + Cz_cr = 0 \tag{1.13}$$

If $x_cz_c \neq 0$, formula (1.12) by virtue of condition (1.9) yields

$$C(B-A)\lambda_z^2 + A(C+B)\lambda_x^2 = 0 \tag{1.14}$$

Formulas (1.12) and (1.14) show that vectors $\mathbf{r}_c$ and λ are orthogonal to the circular cross section of the ellipsoid of inertia of the transformed body at point O. Here $A > B$ and, when $A = B$ or $z_c = \lambda_z = 0$, then $x_c = \lambda_x = 0$; if $x_c = \lambda_x = 0$, then either $A = B$, or $z_c = \lambda_z = 0$.

2. Let us consider the case of development of resonance

$$\omega_3 = \omega_2 \tag{2.1}$$

in a system of linked oscillators defined by Eqs. (1.8).

Formulas (1.12) and (1.14) follow directly from this, and a resonance that satisfies condition (2.1) is defined by a relationship which is independent of components λ_x and λ_z. This is apparent in equalities (1.12) and 1.13).

Note. The above statement can be directly obtained from Eqs. (1.1) without the use of the related linear system. Let

$$-a_1p + a_2r = 0 \quad (a_1^2 + a^2{}_2 \neq 0) \tag{2.2}$$

be the invariant of system (1.1) with conditions (1.3), where a_1 and a_2 are some constant coefficients. Thus, if condition

$$Aa_2l_x + Ca_1l_z = 0 \tag{2.3}$$

is satisfied, then, excluding $p = \text{const}$ (or $r = \text{const}$), in accordance with Eqs. (1.1) for $q \neq 0$ we obtain conditions

$$A(B-A)a_2^2 + C(C+B)a_1^2 = 0 \tag{2.4}$$

$$Aa_2\lambda_x - Ca_1\lambda_z = 0 \tag{2.5}$$

Thus condition (2.4) of existence of the linear invariant (2.2) of system (1.1) obtained with restrictions (1.3) and (2.3) is independent of components λ_x and λ_z. If it is now stipulated that formula (2.2) must simultaneously be an invariant also for system (1.5), we obtain as a corollary $a_1 = a_{31}$, $a_2 = a_{11}$, and conditions (2.4) and (2.5) coincide with formulas (1.12) and (1.9).

3. Let there exist the integral

$$r(t) = r_0 \tag{3.1}$$

For system (1.5) this integral exists for $a_{31} = a_{33} = 0$, which corresponds to conditions

$$\lambda_x = (-Bl_x\gamma_x^\circ)^{1/2} \qquad \lambda_z = -(-Bl_z\gamma_z^\circ)^{1/2} \tag{3.2}$$

It follows from formulas (3.2) that, when λ_x and λ_z are nonzero, then in accordance with (1.4) $x_c < 0$, and $z_c > 0$ for $\gamma_z^\circ < 0$ and $z_c < 0$ for $\gamma_z^\circ > 0$. The conditions of existence of the integral (3.1) for system (1.5) are identically satisfied for $\lambda_x = x_c = 0$. These relationships together with (1.3) constitute an incomplete system of symmetrization conditions of the Lagrange kind.

If the quantities $\Delta_1 = a_{11}\omega_3^2$ and $\Delta_2 = a_{13}\omega_3^2$ are nonzero, there exists the unique transformation

$$p = (-a_{13}P + a_{11}R)\Delta_1^{-1}, \quad r = (a_{13}P + a_{33}R)\Delta_2^{-1} \tag{3.3}$$

inverse of transformation (1.7). If, however, $\Delta_1 = \Delta_2 = 0$, then either $a_{11} = a_{13} = 0$ or $\omega_3 = 0$, and a_{11} and a_{13} are nonzero. In the first case components λ_x and λ_z must satisfy conditions (3.2) and in the second the condition

$$A(\lambda_x^2 + Bl_x\gamma_x^\circ) + C(\lambda_z^2 + Bl_z\gamma_z^\circ) = 0 \tag{3.4}$$

From formula (3.4) with conditions (1.4) and (1.9) we obtain

$$Ax_c^2 + Cz_c^2 = 0$$

hence $x_c = z_c = 0$. This means that gyrostat center of mass is situated on the isotropic cone. Since we consider here only gyrostat motions inside the isotropic cone, we conclude that the second case is not possible because of adopted assumptions. Consequently there is no linear invariant of system (1.5) of the form

$$a_{11}p + a_{13}r = \text{const}$$

which corresponds to $\omega_3 = 0$.

Thus, if Δ_1 and Δ_2 are nonzero, formulas (1.7) and (3.3) determine the homeomorphism in the neighborhood of the transformed body equilibrium position.

Let $x_c = \lambda_x = 0$. Then for $q \neq 0$ either $p(t) = 0$ or $A = B$, and the integral (3.1) of system (1.5) is also the integral of system (1.1). The latter of these cases together with conditions (1.3) defines the gyroscopic analog of the Lagrange kind in the space ${}^1R_3$. Structural conditions for this case can be obtained from the resonance formula (2.1).

Let now $x_c = q(t) = 0$; then (3.1) is the integral of system (1.1), and

$$p^\cdot = k\gamma_y, \qquad p = f^{-1}(l_z\gamma_x - \lambda_x r_0) \tag{3.5}$$
$$k = A^{-1}l_z, \qquad f = \lambda_z + (A + C)r_0 \neq 0$$

Differentiating the last equality with respect to t with allowance for the first of Eqs. (1.2) and comparing the result with formula (3.5) for $p^\cdot$, we obtain

$$\lambda_z + (2A + C)r_0 = 0 \tag{3.6}$$

Cases of degeneration when $z_c = 0$ or $\gamma_y = 0$ are not considered here. Note that the structural condition $2A = C$, which together with previous conditions defines the

Bobylev-Steklov case (in the Bobylev form) for the Euclidean space [14], is not contained in (3. 6). The gyrostatic analog of that case can be, however, found in the space ${}^1R_3$.

In the considered case system (1. 1) has in addition to the integral (3. 1) the integrals of energy and area

$$Ap^2 + Cr_0^2 + 2l_z\gamma_z = h$$

$$(Ap + \lambda_x)\gamma_x + (Cr_0 + \lambda_z)\gamma_z = H$$

where h and H are constants of integration. Substituting in the area integral the expressions for γ_x and γ_z from Eq. (3. 5) and the energy integral, we obtain a formula which becomes an identity when two independent conditions are satisfied. One of these is expressed by (3. 6) and the other is of the form

$$\lambda_x^2 r_0 = Hl_z + Ar_0 (h - Cr_0^2) \qquad (3.7)$$

Substituting into the trivial integral $\gamma^2 = \varkappa$ the expressions for γ_x and γ_z and taking into account condition (3. 6), we obtain

$$\gamma_y^2 = (2l_z)^{-2}[(h - Cr_0^2 - Ap^2)^2 - 4\,(Ap - \lambda_x)^2 r_0^2 - 4\varkappa l_z^2] \qquad (3.8)$$

Equations (3. 5) and the energy integral yield

$$\gamma_x = (Ak)^{-1} r_0 \,(\lambda_x - Ap), \quad \gamma_z = a - (2k)^{-1} p^2 \qquad (3.9)$$

$$kt = \int_{p_0}^{p} \gamma_y^{-1}(s)\, ds, \quad a = (2l_z)^{-1} \cdot (h - Cr_0^2)$$

where λ_x is determined by condition (3. 7) and s is the variable of integration.

Formulas (3. 8) and (3. 9) can be interpreted in the Lobachevski plane using the Beltrami-Klein projective model, as was done in [7]. Such interpretation is possible in the case when all points of the reduced body lie in one of the sheets of the real radius sphere of the space ${}^1R_3$.

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STABILITY OF UNSTEADY MOTIONS IN FIRST APPROXIMATION

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Results obtained in [1] are extended to nonautonomous systems and a wider class of nonlinearities. The question of application of the Liapunov vector function is considered.

1. Let us consider the system of differential equations of perturbed motion

$$\begin{aligned} \mathbf{y}^{\boldsymbol{\cdot}} &= Q(t)\mathbf{x} + R(t)\mathbf{y} + \mathbf{Y}^\circ(t, \mathbf{y}) + \mathbf{Y}(t, \mathbf{x}, \mathbf{y}) \\ \mathbf{x}^{\boldsymbol{\cdot}} &= P(t)\mathbf{x} + \mathbf{X}(t, \mathbf{x}, \mathbf{y}), \quad \mathbf{x} \in \mathbf{R}^n, \quad \mathbf{y} \in \mathbf{R}^k \end{aligned} \tag{1.1}$$

where P, Q and R are continuous and bounded for $t \geqslant 0$ matrices of corresponding order and functions $\mathbf{Y}$ and $\mathbf{X}$ satisfy conditions

$$\mathbf{Y}(t, 0, \mathbf{y}) \equiv 0, \quad \mathbf{X}(t, 0, \mathbf{y}) \equiv 0 \tag{1.2}$$

$$\frac{\|\mathbf{Y}(t, \mathbf{x}, \mathbf{y})\| + \|\mathbf{X}(t, \mathbf{x}, \mathbf{y})\|}{\|\mathbf{x}\|} \underset{t \geqslant 0}{\rightrightarrows} 0 \quad \text{for } \|\mathbf{x}\| + \|\mathbf{y}\| \to 0 \tag{1.3}$$

We assume that solutions of the linear system

$$\mathbf{x}^{*\boldsymbol{\cdot}} = P(t)\mathbf{x}^* \tag{1.4}$$

satisfy the condition

$$\|\mathbf{x}^*(t; t_0, \mathbf{x}_0^*)\| \leqslant B\|\mathbf{x}_0^*\| e^{-\alpha(t-t_0)} \quad (B > 0,\ \alpha > 0\text{-const};\ t \geqslant t_0 \geqslant 0) \tag{1.5}$$

Let us consider the system

$$\mathbf{y}^{*\boldsymbol{\cdot}} = R(t)\mathbf{y}^* + \mathbf{Y}^\circ(t, \mathbf{y}^*) \tag{1.6}$$

which is obtained from the first group of Eqs. (1.1) for $\mathbf{x} = 0$ whose solutions are denoted by $\mathbf{y}^*(t; t_0, \mathbf{y}_0^*)$. The variational equations for system (1.6) are of the form